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DETERMINATION OF THE TWO-COLOR RADO NUMBER FOR $a_1x_1 + \cdots + a_mx_m = x_0$

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Abstract. For positive integers $a_1, a_2, \ldots, a_m$, we determine the least positive integer $R(a_1, \ldots, a_m)$ such that for every 2-coloring of the set $[1, n] = \{1, \ldots, n\}$ with $n \geqslant R(a_1, \ldots, a_m)$ there exists a monochromatic solution to the equation $a_1x_1 + \cdots + a_mx_m = x_0$ with $x_0, \ldots, x_m \in [1, n]$. The precise value of $R(a_1, \ldots, a_m)$ is shown to be $av^2 + v - a$, where $a = \min\{a_1, \ldots, a_m\}$ and $v = \sum_{i=1}^m a_i$. This confirms a conjecture of B. Hopkins and D. Schaal.

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $[a, b] = \{x \in \mathbb{N} : a \leqslant x \leqslant b\}$ for $a, b \in \mathbb{N}$. For $k, n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, we call a function $\Delta : [1, n] \to [0, k-1]$ a *k-coloring* of the set $[1, n]$, and $\Delta(i)$ the *color* of $i \in [1, n]$. Given a k-coloring of the set $[1, n]$, a solution to the linear diophantine equation

$$a_0x_0 + a_1x_1 + \cdots + a_mx_m = 0 \quad (a_0, a_1, \ldots, a_m \in \mathbb{Z})$$

with $x_0, x_1, \ldots, x_m \in [1, n]$ is called *monochromatic* if $\Delta(x_0) = \Delta(x_1) = \cdots = \Delta(x_m)$.

Let $k \in \mathbb{Z}^+$. In 1916, I. Schur [S] proved that if $n \in \mathbb{Z}^+$ is sufficiently large then for every *k-coloring* of the set $[1, n]$, there exists a monochromatic solution to

$$x_1 + x_2 = x_0$$

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with $x_0, x_1, x_2 \in [1, n]$.

Let $k \in \mathbb{Z}^+$ and $a_0, a_1, \ldots, a_m \in \mathbb{Z} \setminus \{0\}$. Provided that $\sum_{i \in I} a_i = 0$ for some $\emptyset \neq I \subseteq \{0, 1, \ldots, m\}$, R. Rado showed that for sufficiently large $n \in \mathbb{Z}^+$ the equation $a_0 x_0 + a_1 x_1 + \cdots + a_m x_m = 0$ always has a monochromatic solution when a k-coloring of $[1, n]$ is given; the least value of such an n is called the *k-color Rado number* for the equation. Since $-1 + 1 = 0$, Schur's theorem is a particular case of Rado's result. The reader may consult the book [LR] by B. M. Landman and A. Robertson for a survey of results on Rado numbers.

In this paper, we are interested in precise values of 2-color Rado numbers. By a theorem of Rado [R], if $a_0, a_1, \ldots, a_m \in \mathbb{Z}$ contain both positive and negative integers and at least three of them are nonzero, then the homogeneous linear equation

$$a_0 x_0 + a_1 x_1 + \cdots + a_m x_m = 0$$

has a monochromatic solution with $x_0, \ldots, x_m \in [1, n]$ for any sufficiently large $n \in \mathbb{Z}^+$ and a 2-coloring of $[1, n]$. In particular, if $a_1, \ldots, a_m \in \mathbb{Z}^+$ $(m \geqslant 2)$ then there is a least positive integer $n_0 = R(a_1, \ldots, a_m)$ such that for any $n \geqslant n_0$ and a 2-coloring of $[1, n]$ the diophantine equation

$$a_1 x_1 + \cdots + a_m x_m = x_0 \tag{1.0}$$

always has a monochromatic solution with $x_0, \ldots, x_m \in [1, n]$.

In 1982, A. Beutelspacher and W. Brestovansky [BB] proved that the 2-color Rado number $R(1, \ldots, 1)$ for the equation $x_1 + \cdots + x_m = x_0$ $(m \geqslant 2)$ is $m^2 + m - 1$. In 1991, H. L. Abbott [A] extended this by showing that for the equation

$$a(x_1 + \cdots + x_m) = x_0 \quad (a \in \mathbb{Z}^+ \text{ and } m \geqslant 2)$$

the corresponding 2-color Rado number $R(a, \ldots, a)$ is $a^3 m^2 + am - a$; that $R(a, \ldots, a) \geqslant a^3 m^2 + am - a$ was first obtained by L. Funar [F], who conjectured the equality. In 2001, S. Jones and D. Schaal [JS] proved that if $a_1, \ldots, a_m \in \mathbb{Z}^+$ $(m \geqslant 2)$ and $\min\{a_1, \ldots, a_m\} = 1$ then $R(a_1, \ldots, a_m) = b^2 + 3b + 1$ where $b = a_1 + \cdots + a_m - 1$; this result actually appeared earlier in Funar [F].

In 2005 B. Hopkins and D. Schaal [HS] showed the following result.

Theorem 1.0. *Let $m \geqslant 2$ be an integer and let $a_1, \ldots, a_m \in \mathbb{Z}^+$. Then*

$$R(a, b) \geqslant R(a_1, \ldots, a_m) \geqslant a(a + b)^2 + b, \tag{1.1}$$

where

$$a = \min\{a_1, \ldots, a_m\} \quad \text{and} \quad b = \sum_{i=1}^{m} a_i - a. \tag{1.2}$$

Hopkins and Schaal ([HS]) conjectured further that the two inequalities in (1.1) are actually equalities and verified this in the case $a = 2$.

In this paper we confirm the conjecture of Hopkins and Schaal; namely, we establish the following theorem.

Theorem 1.1. *Let $m \geqslant 2$ be an integer and let $a_1, \ldots, a_m \in \mathbb{Z}^+$. Then*

$$R(a_1, \ldots, a_m) = a(a + b)^2 + b, \tag{1.3}$$

where a and b are as in (1.2)*.*

By Theorem 1.1, if $a_1, \ldots, a_m \in \mathbb{Z}^+$ and $n \geqslant av^2 + v - a$ with $a = \min\{a_1, \ldots, a_m\}$ and $v = a_1 + \cdots + a_m$, then for any $X \subseteq [1, n]$ either there are $x_1, \ldots, x_m \in X$ such that $\sum_{i=1}^m a_ix_i \in X$ or there are $x_1, \ldots, x_m \in [1, n] \setminus X$ such that $\sum_{i=1}^m a_ix_i \in [1, n] \setminus X$.

In the next section we reduce Theorem 1.1 to the following weaker version.

Theorem 1.2. *Let $a, b, n \in \mathbb{Z}^+$, $a \leqslant b$ and $n \geqslant av^2 + b$ with $v = a + b$. Suppose that $b(b - 1) \not\equiv 0 \pmod{a}$ and $\Delta : [1, n] \to [0, 1]$ is a 2-coloring of $[1, n]$ with $\Delta(1) = 0$ and $\Delta(a) = \Delta(b) = \delta \in [0, 1]$. Then there is a monochromatic solution to the equation*

$$ax + by = z \qquad (x, y, z \in [1, n]). \tag{1.4}$$

In Sections 3 and 4 we will prove Theorem 1.2 in the cases $\delta = 0$ and $\delta = 1$ respectively.

2. Reduction of Theorem 1.1 to Theorem 1.2

Let us first give a key lemma which will be used in Sections 2–4.

Lemma 2.1. *Let $k, l, n \in \mathbb{Z}^+$ with $l < n$, and let $\Delta : [1, n] \to [0, 1]$ be a 2-coloring of $[1, n]$. Suppose that $kx + ly = z$ has no monochromatic solution with $x, y, z \in [1, n]$. Assume also that u is an element of $[1, n - l]$ with $\Delta(u) = \delta$ and $\Delta(u + l) = 1 - \delta$.*

(i) *If $w \in \mathbb{Z}^+$, $w \leqslant (n - ku)/l$ and $\Delta(w) = \delta$, then $\Delta(w - hk) = \delta$ whenever $h \in \mathbb{N}$ and $w - hk > 0$.*

(ii) *If $w \in [1, n]$ and $\Delta(w) = 1 - \delta$, then $\Delta(w + hk) = 1 - \delta$ whenever $h \in \mathbb{N}$ and $w + hk \leqslant (n - ku)/l$.*

Proof. It suffices to handle the case $h = 1$, since we can consider $w \mp (h - 1)k$ instead of w if $h > 1$.

(i) As $\Delta(u) = \Delta(w) = \delta$ and $w \leqslant (n - ku)/l$, we have $\Delta(ku + lw) = 1 - \delta$. By $\Delta(u + l) = 1 - \delta$ and $k(u + l) + l(w - k) = ku + lw$, if $w - k > 0$ then $\Delta(w - k) = \delta$.

(ii) Since $\Delta(u+l) = \Delta(w) = 1-\delta$ and $(w+k)l + ku \leqslant n$, we have $\Delta(k(u+l)+lw) = \delta$. Note that $\Delta(u) = \delta$ and $ku+l(w+k) = k(u+l)+lw$. So $\Delta(w+k) = 1-\delta$.

The proof of Lemma 2.1 is now complete. $\square$

Now we deduce Theorem 1.1 from Theorem 1.2.

Proof of Theorem 1.1. By Theorem 1.0, it suffices to show that $R(a,b) \leqslant av^2 + b$, where $v = a+b$. Since $m \geqslant 2$, we have $a \leqslant b$.

Let $n \geqslant av^2+b$ be an integer and let $\Delta : [1,n] \to [0,1]$ be a 2-coloring of $[1,n]$. Without loss of generality, we may assume that $\Delta(1) = 0$. Suppose, for contradiction, that there doesn't exist any monochromatic solution to the equation (1.4).

Since $a \cdot 1 + b \cdot 1 = v$, we have $\Delta(v) \neq \Delta(1) = 0$, and hence

$$\Delta(v) = 1. \tag{2.1}$$

Similarly, as $av + bv = v^2$, we must have

$$\Delta(v^2) = 0 \quad \text{and} \quad \Delta(av^2 + b \cdot 1) = 1. \tag{2.2}$$

Claim 2.1. $\Delta(a) = \Delta(b) \neq \Delta(av) = \Delta(bv)$.

As $aa + ba = av$ and $ab + bb = bv$, we have

$$\Delta(av) \neq \Delta(a) \quad \text{and} \quad \Delta(bv) \neq \Delta(b).$$

If $\Delta(a) \neq \Delta(b)$, then

$$\Delta(av) = \Delta(b) \neq \Delta(a) = \Delta(bv)$$

and hence

$$\Delta(a) = \Delta(ab + b(av)) = \Delta(abv + ab) = \Delta(a(bv) + ba) = \Delta(b),$$

which contradicts $\Delta(a) \neq \Delta(b)$.

Below, we let $\delta = \Delta(a) = \Delta(b)$ and hence $\Delta(av) = \Delta(bv) = 1-\delta$.

In view of Claim 2.1 and Theorem 1.2, a divides $b(b-1)$ since (1.4) has no monochromatic solution.

Claim 2.2. $\Delta(ab + bv + (1-\delta)av) = 0$.

Recall that $\Delta(v) = 1$ by (2.1). If $\delta = 1$, then $\Delta(b) = 1 = \Delta(v)$, and hence $\Delta(ab+bv) = 0$. When $\delta = 0$, we have $\Delta(b) = 0 < \Delta(b+a) = \Delta(v) = 1$, and hence $\Delta(v+b) = 1$ by Lemma 2.1(ii) (with $k = u = b$, $l = a$ and $w = v$) since $v + b = a + 2b \leqslant (n-b^2)/a$; therefore, $\Delta(a(v+b) + bv) = 0$. This completes the proof of Claim 2.2.

Observe that

$$ab(v-1)+ab+b+(1-\delta)av \leqslant abv+b+av \leqslant av^2+b \leqslant n.$$

Claim 2.3. For every $i = 1, \ldots, a$ we have

$$\Delta(ib(v-1)+ab+b+(1-\delta)av) = 0. \tag{2.3}$$

When $i = 1$, (2.3) holds by Claim 2.2. Now let $1 < i \leqslant a$ and assume that (2.3) holds with i replaced by $i-1$. Then

$$\begin{aligned}&\Delta\left(a\left((i-1)\frac{b(b-1)}{a}+ib+(1-\delta)v\right)+b\cdot 1\right)\\ =&\Delta((i-1)b(v-1)+ab+b+(1-\delta)av) = 0 = \Delta(1)\end{aligned}$$

by the induction hypothesis. Therefore,

$$\Delta\left((i-1)\frac{b(b-1)}{a}+ib+(1-\delta)v\right) = 1 = \Delta(v),$$

and hence

$$\begin{aligned}&\Delta(ib(v-1)+ab+b+(1-\delta)av)\\ =&\Delta\left(a\left((i-1)\frac{b(b-1)}{a}+ib+(1-\delta)v\right)+bv\right) = 0.\end{aligned}$$

This concludes the induction proof of Claim 2.3.

Putting $i = a$ in (2.3) we find that

$$\Delta\left(abv+b+(1-\delta)av\right) = 0 = \Delta(1).$$

If $\delta = 1$, then $\Delta(a(bv)+b\cdot 1) = 0 = \Delta(1)$, and hence $\Delta(bv) = 1 = \Delta(b)$, which is impossible by Claim 2.1. Thus $\delta = 0$ and

$$\Delta(aa+b(av+a+1)) = \Delta(abv+b+av) = 0 = \Delta(a).$$

It follows that $\Delta(av+a+1) = 1$. Also, if $a = 1$ then $\Delta(av^2+b) = \Delta(abv+b+av) = 0$. Since $\Delta(av^2+b) = 1$ by (2.2), and

$$a(av-b)+b(av+a+1) = av^2+b,$$

we must have $a \geqslant 2$ and $\Delta(av-b) = 0$. As $\Delta(b) = 0 < \Delta(b+a) = 1$ and

$$av-b = v^2-b(v+1) < v^2-b(b-1) \leqslant v^2-\frac{b(b-1)}{a} \leqslant \frac{n-b^2}{a},$$

we have $\Delta(a^2+b) = \Delta(av-b-(a-2)b) = 0$ by Lemma 2.1(i) with $k = u = b$, $l = a$ and $w = av-b$. However, $\Delta(a^2+b) = \Delta(aa+b\cdot 1) = 1$ since $\Delta(a) = 0 = \Delta(1)$, so we get a contradiction. This completes the proof. $\square$

3. Proof of Theorem 1.2 with $\delta = 0$

To prove Theorem 1.2 in the case $\delta = 0$, we should deduce a contradiction under the assumption that (1.4) has no monochromatic solution. Recall the condition $\Delta(1) = \Delta(a) = \Delta(b) = 0$. It is clear that $\Delta(a \cdot 1 + b \cdot 1) \neq \Delta(1) = 0$.

Note that $a(v-1) + b(v-1) = v^2 - v \leqslant av^2 + b \leqslant n$. We make the following claim first.

Claim 3.1. $\Delta(ai + bj) = 1$ for any $i, j \in [1, a]$.

Since $\Delta(a) = 0 < \Delta(a+b) = 1$ and

$$v + (i-1)a \leqslant a^2 + b \leqslant \frac{ab^2 + 2a^2b + a^3 - a^2}{b} < \frac{av^2 + b - a^2}{b} \leqslant \frac{n - a^2}{b},$$

we have $\Delta(ai + b) = 1$ by Lemma 2.1(ii) with $k = u = a$, $l = b$ and $w = v$. Similarly, as

$$(ai + b) + b(j-1) \leqslant a^2 + ab = av = v^2 - bv < v^2 - b\frac{b-1}{a} \leqslant \frac{n - b^2}{a},$$

by Lemma 2.1(ii) with $k = u = b$, $l = a$ and $w = ai + b$ we get that

$$\Delta(ai + bj) = \Delta(ai + b + b(j-1)) = \Delta(ai + b) = 1.$$

This proves Claim 3.1.

Claim 3.2. $\Delta(c) = 0$ for any $c \in [1, v-1]$.

Suppose that $c \in [b+1, v-1]$ and $\Delta(c) = 1$. Then $\Delta(av+bc) = 0 = \Delta(a)$ since $\Delta(v) = 1 = \Delta(c)$. Therefore,

$$\Delta(a(av + bc) + ba) = 1.$$

Clearly,

$$a(av + bc) + ba = a(a^2 + b(c - b + 1)) + b(a^2 + ba),$$

and $\Delta(a^2 + b(c - b + 1)) = 1 = \Delta(a^2 + ba)$ by Claim 3.1. Thus we get a monochromatic solution to (1.4), contradicting our assumption. So, $\Delta(c) = 0$ for all $c \in [b+1, v-1]$.

Now let $c \in [1, b]$. Then there is $\bar{c} \in [b, v-1]$ such that $\bar{c} - c = ha$ for some $h \in \mathbb{N}$ (e.g., $\bar{c} = c$ when $c = b$). Recall that $\Delta(a) = 0 < \Delta(a+b) = \Delta(v) = 1$ and also $\Delta(b) = 0$. As $\Delta(\bar{c}) = 0$ and

$$\bar{c} < v < \frac{v^2 - a}{b} < \frac{av^2 + b - a^2}{b} \leqslant \frac{n - a^2}{b},$$

by Lemma 2.1(i) with $k = u = a$, $l = b$ and $w = \bar{c}$, we have $\Delta(c) = \Delta(\bar{c} - ha) = 0$. This concludes the proof of Claim 3.2.

Claim 3.3. $\Delta(ai+bj)=1$ for any $i,j\in[1,v-1]$.

By Claim 3.2 we have $\Delta(i)=\Delta(j)=0$. Thus $\Delta(ai+bj)=1$ since (1.4) has no monochromatic solution. So Claim 3.3 holds.

Let d be the greatest common divisor of a and b. Since $a\nmid b$, we have $d<a<b$, hence both $a'=a/d$ and $b'=b/d$ are greater than one. By elementary number theory, there is $s\in[1,b'-1]$ such that $a's\equiv 1 \pmod{b'}$. Since $1<a's<a'b'$, we have $t=(a's-1)/b'\in[1,a'-1]$ and $b't<a'b'\leqslant av$. Observe that

$$a(av+b's)+b(av-b't)=av(a+b)+b'd=av^2+b\leqslant n.$$

As $\Delta(v^2)=\Delta(av+bv)\neq\Delta(v)=1$, we have $\Delta(v^2)=0=\Delta(1)$ and hence $\Delta(av^2+b\cdot 1)=1$. Therefore,

$$\Delta(av+b's)=0 \quad\text{or}\quad \Delta(av-b't)=0. \tag{3.1}$$

Since $a+s, a-t\in[1,v-1]$, we have

$$\Delta(av+bs)=\Delta(a^2+b(a+s))=1=\Delta(a^2+b(a-t))=\Delta(av-bt)$$

by Claim 3.3, which contradicts (3.1) if $b=b'$. So $b'\neq b$, and hence $d>1$.

In view of (3.1), we distinguish two cases.

Case 3.1. $\Delta(av+b's)=0$.

Choose $s_1\in\mathbb{Z}^+$ such that $1\leqslant as_1-b't\leqslant a$. Since $as_1\leqslant a+b't\leqslant a+b(a-1)\leqslant ab$, we have $s_1\leqslant b$. Clearly, $\Delta(aa+ba)=\Delta(as_1+b\cdot 1)=1$ by Claim 3.3, and

$$a(a^2+ab)+b(as_1+b)\leqslant a^2v+b(a+ba)\leqslant av^2+b\leqslant n.$$

Therefore,

$$\Delta(a(a^2+ab)+b(as_1+b))=0.$$

However,

$$a(a^2+ab)+b(as_1+b)=a(av+b's)+b(as_1-b't+b-1)$$

and $\Delta(as_1-b't+b-1)=0$ by Claim 3.2. This contradicts the assumption that (1.4) has no monochromatic solution.

Case 3.2. $\Delta(av-b't)=0$.

Choose $s_2\in\mathbb{Z}$ so that $0\leqslant a't-as_2\leqslant a-1$. Clearly, $0\leqslant s_2\leqslant t\leqslant a'-1<a-1$. With the help of Claim 3.2, $\Delta(a't-as_2+b)=0=\Delta(av-b't)$. Since

$$a(av-b't)+b(a't-as_2+b)=a^2v-abs_2+b^2\leqslant av^2+b\leqslant n,$$

we have $\Delta(a^2v-abs_2+b^2)=1$. Observe that

$$a^2v-abs_2+b^2=a(a^2+b)+b(a(a-1-s_2)+b)$$

and $\Delta(a^2+b)=\Delta(a(a-1-s_2)+b)=1$ by Claim 3.3. So we get a monochromatic solution to (1.4), contradicting our assumption.

4. Proof of Theorem 1.2 with $\delta = 1$

Assume the conditions of Theorem 1.2 with $\delta = 1$, and that (1.4) doesn't have a monochromatic solution. Our goal is to deduce a contradiction.

Since $\Delta(a) = \Delta(b) = \delta = 1$, $av = aa + ba$ and $bv = ab + bb$, we have

$$\Delta(av) = \Delta(bv) = 0. \tag{4.1}$$

Thus there is a positive multiple $u_1 \leqslant b(v-1)$ of b such that $\Delta(u_1) = 1$ and $\Delta(u_1 + b) = 0$; also there is a positive multiple $u_2 \leqslant a(v-1)$ of a such that $\Delta(u_2) = 1$ and $\Delta(u_2 + a) = 0$.

Observe that

$$a^2 + a + 1 < a^2 \cdot \frac{v}{b} + a + 1 = \frac{(av^2 + b) - ab(v-1)}{b} \leqslant \frac{n - au_1}{b}.$$

As $\Delta(1) = 0$ and $1 + a < a^2 + a + 1$, we have $\Delta(1 + a) = 0$ by Lemma 2.1(ii) with $k = a$, $l = b$, $u = u_1$ and $w = 1$. Thus,

$$\Delta(av + v) = \Delta(a(a+1) + b(a+1)) = 1. \tag{4.2}$$

Claim 4.1. $\Delta(a^2 + a) = 1 \Rightarrow \Delta(a) = \Delta(2a) = \cdots = \Delta(a^2) = 1$.

Recall that $\Delta(u_1) = 1 > \Delta(u_1 + b) = 0$ and $a^2 + a < (n - au_1)/b$. By Lemma 2.1(i) with $k = a$, $l = b$, $u = u_1$ and $w = a^2 + a$, if $\Delta(a^2 + a) = 1$ then $\Delta(a^2 + a - ha) = 1$ for all $h = 0, \ldots, a$. This proves Claim 4.1.

Claim 4.2. For $w \in [1, n]$ and $h \in \mathbb{N}$ with $w + hb \leqslant av + b$, we have $\Delta(w) = 0 \Rightarrow \Delta(w + hb) = 0$.

Note that

$$av + b < av + b + \frac{b}{a} = \frac{(av^2 + b) - ab(v-1)}{a} \leqslant \frac{n - bu_2}{a}.$$

So we get Claim 4.2 by applying Lemma 2.1(ii) with $k = b$, $l = a$ and $u = u_2$.

Write $b = aq + r$ with $q, r \in \mathbb{N}$ and $r < a$. Since $a \leqslant b$ and $a \nmid b(b-1)$, we have $q \geqslant 1$ and $r \geqslant 2$.

Claim 4.3. $\Delta(r) = 0 \Longrightarrow \Delta(a^2) = 0$.

Assume that $\Delta(r) = 0$. As $\Delta(r + aq) = \Delta(b) = 1$, there is $u_3 \in \{r, r + a, \ldots, r + a(q-1)\}$ such that $\Delta(u_3) = 0$ and $\Delta(u_3 + a) = 1$. Since $\Delta(av) = 0$ (cf. (4.1)) and

$$av = v^2 - bv < v^2 - b^2 < v^2 + b - b(b-1) \leqslant \frac{(av^2 + b) - b(b-a)}{a} \leqslant \frac{n - bu_3}{a},$$

we have $\Delta(a^2) = \Delta(av - ab) = 0$ by Lemma 2.1(i) with $k = b$, $l = a$, $u = u_3$ and $w = av$.

Claim 4.4. $\Delta(r) = \Delta(ar) = 1 \Longrightarrow \Delta(av + a) = 0$.

Suppose that $\Delta(r) = \Delta(ar) = 1$. Then $\Delta(vr) = \Delta(ar + br) = 0$. So there is $u_4 \in \{ar, ar + b, \dots, ar + (r - 1)b\}$ such that $\Delta(u_4) = 1$ and $\Delta(u_4 + b) = 0$. Since $\Delta(av) = 0$ by (4.1), and

$$av + a < av + a(a-r)\frac{v}{b} = av\frac{v-r}{b} < \frac{(av^2 + b) - a(vr - b)}{b} \leqslant \frac{n - au_4}{b},$$

we have $\Delta(av + a) = 0$ by Lemma 2.1(ii) with $k = a$, $l = b$, $u = u_4$ and $w = av$.

Claim 4.5. $\Delta(av + a) = 0$.

Clearly, $(a^2 + a) + ab = av + a \leqslant av + b$. If $\Delta(a^2 + a) = 0$, then we have $\Delta(av + a) = \Delta((a^2 + a) + ab) = 0$ by applying Claim 4.2 with $w = a^2 + a$ and $h = a$. In the case $\Delta(a^2 + a) = 1$, by Claim 4.1 we have $\Delta(a^2) = 1 = \Delta(ar)$, hence $\Delta(r) = \Delta(ar) = 1$ by Claim 4.3 and $\Delta(av + a) = 0$ by Claim 4.4.

Claim 4.6. There exists $u \in [1, ab - a]$ such that $\Delta(u) = 1$ and $\Delta(u + a) = 0$.

As a does not divide b, the greatest common divisor d of a and b is smaller than a, and hence $1 < a' = a/d < b' = b/d$. If $\Delta(db) = 0$, then we have

$$\Delta(ab) = \Delta(db + (a - d)b) = 0 < 1 = \Delta(a)$$

by applying Claim 4.2 with $w = db$ and $h = a - d$, hence there is $u \in \{a, 2a, \dots, (b-1)a\}$ such that $\Delta(u) = 1$ and $\Delta(u+a) = 0$. Below we work under the condition $\Delta(db) = 1$.

Case 4.1. $\Delta(d) = 1$.

In this case, $d > 1$ since $\Delta(d) \neq \Delta(1)$. Note that $\Delta(dv) = \Delta(ad+bd) = 1-\Delta(d) = 0$. As $\Delta(db) = 1$, for some $u \in \{db, db+a, \dots, db+(d-1)a\}$ we have $\Delta(u) = 1 > \Delta(u+a) = 0$. Note that $a'b'-a'-b' = (a'-1)(b'-1)-1 \geqslant 0$ and

$$u \leqslant dv - a = d^2(a' + b') - a \leqslant d^2a'b' - a = ab - a.$$

Case 4.2. $\Delta(d) = 0$.

Choose $s \in [0, b - 1]$ such that $as \equiv d \pmod b$. Clearly $s \neq 0, 1$ since $d < a < b$. For $t = (as - d)/b$, we have $0 < t < a$. As $\Delta(d) = 0$ and $d + bt = as < ab \leqslant av + b$, we have $\Delta(as) = \Delta(d + bt) = 0$ by Claim 4.2 with $w = d$ and $h = t$. Recall that $\Delta(a) = 1$. So there is $u \in \{a, 2a, \dots, (s - 1)a\}$ such that $\Delta(u) = 1$ and $\Delta(u + a) = 0$. Clearly, $u \leqslant (s - 1)a < ab - a$. This concludes the proof of Claim 4.6.

Let u be as required in Claim 4.6. Then

$$av + v = v^2 - (b - 1)v \leqslant v^2 - b(b - 1) < \frac{(av^2 + b) - b(ab - a)}{a} \leqslant \frac{n - bu}{a}.$$

Recall that $\Delta(av+v)=1$ by (4.2). Thus $\Delta(av+a)=\Delta((av+v)-b)=1$ by Lemma 2.1(i) with $k=b$, $l=a$, and $w=av+v$. This contradicts Claim 4.5 and we are done.

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